The universal property of $\mathcal{D}^{-}(\mathcal{A})$

based on Section 1.3.3 of Higher Algebra by Jacob Lurie

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Conventions

Contrary to the conventions of Higher Algebra, we will omit the nerve construction of 1-categories from our notation.

We will often decorate functor categories with some symbols to mean a full subcategory spanned by functors satisfying some property. Examples:

- Fun^{rex}: subcategory of right exact functors (i. e. those which preserve finite colimits).
- $\mathrm{Fun}^{|-|,\amalg}$: subcategory of functors which preserve geometric realizations and finite coproducts.

We fix an abelian category $\mathcal{A}$ with enough projective objects. Changing universes if necessary, we assume that $\mathcal{A}$ is small.

The main theorem

Definition (1.3.3.1)

Let $\mathcal{C}$ and $\mathcal{C}'$ be stable ∞-categories with t-structures.
A functor $F\colon \mathcal{C} \to \mathcal{C}'$ is called *right t-exact (rtex)* if it is exact (i. e. preserves finite limits and colimits) and $F(\mathcal{C}_{\geqslant 0}) \subseteq \mathcal{C}'_{\geqslant 0}$.

Theorem (1.3.3.2)

Let $\mathcal{C}$ be a stable ∞-category equipped with a left complete t-structure.
Then

$$\begin{aligned}\mathrm{Fun}^{rtex, \mathcal{A}_{proj} \mapsto \mathcal{C}^{\heartsuit}}(\mathcal{D}^{-}(\mathcal{A}), \mathcal{C}) &\to \mathrm{Fun}^{rex_{ab}}(\mathcal{A}, \mathcal{C}^{\heartsuit}) \\ F &\mapsto \tau_{\leqslant 0} \circ (F|\mathcal{D}^{-}(\mathcal{A})^{\heartsuit})\end{aligned}$$

is an equivalence.

A proof sketch for the main theorem

$$\mathrm{Fun}^{\mathrm{rtex},\mathcal{A}_{\mathrm{proj}}\mapsto\mathcal{C}^{\heartsuit}}(\mathcal{D}^{-}(\mathcal{A}),\mathcal{C})$$

$\updownarrow$ 1.3.3.11, 1.3.3.16

$$\mathrm{Fun}^{|-|,\amalg,\mathcal{A}_{\mathrm{proj}}\mapsto\mathcal{C}^{\heartsuit}}(\mathcal{D}^{-}_{\geqslant 0}(\mathcal{A}),\mathcal{C}_{\geqslant 0}) \xleftrightarrow{\text{1.3.3.8}} \mathrm{Fun}^{\amalg}(\mathcal{A}_{\mathrm{proj}},\mathcal{C}^{\heartsuit})$$

$\updownarrow$ 1.3.3.9 "projective resolutions"

$$\mathrm{Fun}^{\mathrm{rex}_{\mathrm{ab}}}(\mathcal{A},\mathcal{C}^{\heartsuit})$$

A lemma about geometric realizations

Lemma (1.3.3.10)

1. *If an ∞-category $\mathcal{C}$ admits finite coproducts and geometric realizations of simplicial objects, then it admits all finite colimits. The converse is true if $\mathcal{C}$ is an n-category for some $n \in \mathbb{N}$.*
2. *If $F : \mathcal{C} \to \mathcal{D}$ is a functor between ∞-categories admitting finite coproducts and geometric realizations of simplicial objects that preserves finite coproducts and geometric realizations, then F is right exact. The converse is true if $\mathcal{C}$ and $\mathcal{D}$ are n-categories for some $n \in \mathbb{N}$.*

Proof of the geometric realization lemma

Proof sketch.

In order to construct all finite colimits from finite coproducts and geometric realizations, it is enough to construct coequalizers.
Note there is a (non-full) inclusion $\iota\colon ([1] \rightrightarrows [0]) \hookrightarrow \Delta^{\mathrm{op}}$ of the coequalizer shape into Δ^{op}. Moreover, using the pointwise formula for Kan extensions, one can show that $\iota_!$ exists because finite coproducts exist.
Hence $\mathrm{colim}_{[1]\rightrightarrows[0]} \simeq \mathrm{colim}_{\Delta^{\mathrm{op}}} \circ \iota_!$ exists.

For the part about n-categories, one shows that $\mathrm{colim}_{\Delta^{\mathrm{op}}} \simeq \mathrm{colim}_{\Delta^{\mathrm{op}}_{\leqslant n}} \circ \iota^*_{\leqslant n}$ using some connectivity estimates after applying a Yoneda embedding and notes that the latter is a finite colimit. □

From right t-exact functors to right exact functors

Lemma (1.3.3.11)

Let $\mathcal{C}$, $\mathcal{C}'$ be stable ∞-categories equipped with t-structures. Then

1. *If $\mathcal{C}$ is right bounded, then restriction along $\mathcal{C}_{\geqslant 0} \hookrightarrow \mathcal{C}$ induces an equivalence $\mathrm{Fun}^{rtex}(\mathcal{C}, \mathcal{C}') \simeq \mathrm{Fun}^{rex}(\mathcal{C}_{\geqslant 0}, \mathcal{C}'_{\geqslant 0})$.*
2. *If $\mathcal{C}$ and $\mathcal{C}'$ are left complete, then $\mathcal{C}_{\geqslant 0}$ and $\mathcal{C}'_{\geqslant 0}$ admit geometric realizations of simplicial objects, and a functor $F \colon \mathcal{C}_{\geqslant 0} \to \mathcal{C}'_{\geqslant 0}$ is right exact if and only if it preserves finite coproducts and geometric realizations.*

Proof the "rtex to rex" lemma, part 1

Proof sketch for (1).

Since $\mathcal{C} = \bigcup_n \mathcal{C}_{\geqslant -n}$ by right boundedness, we have

$$\mathrm{Fun}^{rtex}(\mathcal{C}, \mathcal{C}') \simeq \lim(\ldots \to \mathrm{Fun}^{rex}(\mathcal{C}_{\geqslant -1}, \mathcal{C}'_{\geqslant -1}) \to \mathrm{Fun}^{rex}(\mathcal{C}_{\geqslant 0}, \mathcal{C}'_{\geqslant 0})).$$

Now the tower on the RHS is essentially constant as $\mathrm{Fun}^{rex}(\mathcal{C}_{\geqslant -n-1}, \mathcal{C}'_{\geqslant -n-1}) \to \mathrm{Fun}^{rex}(\mathcal{C}_{\geqslant n}, \mathcal{C}'_{\geqslant n})$ has a (homotopy) inverse given by "conjugation by Σ". □

From right t-exact functors to right exact functors

Lemma (1.3.3.11)

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Proof the "rtex to rex" lemma, part 2

Proof sketch for (2).

Left completeness means that

$$\mathcal{C}_{\geqslant 0} \simeq \lim(\ldots \to (\mathcal{C}_{\geqslant 0})_{\leqslant 1} \to (\mathcal{C}_{\geqslant 0})_{\leqslant 0}).$$

Moreover, each $(\mathcal{C}_{\geqslant 0})_{\leqslant n}$ is an finitely cocomplete $(n+1)$-category. Hence, by Lemma 1.3.3.10 about geometric realizations, they all admit geometric realizations. Those are preserved by the truncation functors in the tower above, so $\mathcal{C}$ admits geometric realizations too.

The statement about preservation of colimits follows by a similar argument reducing the statement to the case of k-categories by virtue of the description of $\mathcal{C}_{\geqslant 0}$ as a limit of such. □

The t-structure on $\mathcal{D}^-(\mathcal{A})$

Proposition (1.3.3.16)

The standard t-structure on $\mathcal{D}^-(\mathcal{A})$ is right bounded and left complete.

Remark

Left completeness is ultimately reduced to the convergence of Postnikov towers of spaces by embedding $\mathcal{D}^-(\mathcal{A})$ into the derived category of a presheaf category which we will be described in the next slide.

A model for Ind-objects

Proposition (1.3.3.13)

Ind$(\mathcal{A})$ *can be identified with* $\mathcal{A}^\wedge := \mathsf{Fun}^\times(\mathcal{A}^{\mathrm{op}}_{proj}, \mathrm{Set})$ *which is again an abelian category with enough projective objects.*

Remark (1.3.3.15)

$\mathcal{D}^-(\mathcal{A})$ *can be identified with the full subcategory of* $\mathcal{D}^-(\mathcal{A}^\wedge)$ *consisting of objects whose homology belongs to (the Yoneda image of)* $\mathcal{A}$*.*

Product preserving presheaves

Proposition (1.3.3.14)

We have equivalences

$$\begin{array}{ccc} \mathcal{D}^-_{\geqslant 0}(\mathcal{A}^\wedge) \xleftrightarrow[\textit{1.3.2.22}]{} \mathcal{N}(\underline{\mathsf{Fun}}(\Delta^{\mathrm{op}}, \mathcal{A}^\wedge)) & & \\ \qquad\qquad\qquad\qquad \updownarrow \textit{Dold–Kan} & & \\ \mathcal{N}(\underline{\mathsf{Fun}}^\times(\mathcal{A}^{\mathrm{op}}_{proj}, \mathrm{sSet})) & \xleftrightarrow[\textit{HTT 5.5.9.3}]{} & \mathsf{Fun}^\times(\mathcal{A}^{\mathrm{op}}_{proj}, \mathcal{S}). \end{array}$$

Moreover, the restriction of the composite along the inclusion $\mathcal{A}_{proj} \hookrightarrow \mathcal{D}^-(\mathcal{A}^\wedge)$ corresponds to the Yoneda embedding.

A proof sketch for the main theorem

$$\mathrm{Fun}^{\mathrm{rtex},\mathcal{A}_{\mathrm{proj}}\mapsto\mathcal{C}^{\heartsuit}}(\mathcal{D}^{-}(\mathcal{A}),\mathcal{C})$$

1.3.3.11, 1.3.3.16 $\updownarrow$

$$\mathrm{Fun}^{|-|,\amalg,\mathcal{A}_{\mathrm{proj}}\mapsto\mathcal{C}^{\heartsuit}}(\mathcal{D}^{-}_{\geqslant 0}(\mathcal{A}),\mathcal{C}_{\geqslant 0}) \xleftrightarrow{\text{1.3.3.8}} \mathrm{Fun}^{\amalg}(\mathcal{A}_{\mathrm{proj}},\mathcal{C}^{\heartsuit})$$

$\updownarrow$ 1.3.3.9 "projective resolutions"

$$\mathrm{Fun}^{\mathrm{rex}_{\mathrm{ab}}}(\mathcal{A},\mathcal{C}^{\heartsuit})$$

A more precise characterization

We still need to prove:

Theorem (1.3.3.8)

Let $\mathcal{C}$ be an ∞-category admitting geometric realizations of simplicial objects.
Then

1. *The restriction functor* $\mathrm{Fun}^{|-|}(\mathcal{D}^{-}_{\geqslant 0}(\mathcal{A}), \mathcal{C}) \to \mathrm{Fun}(\mathcal{A}_{proj}, \mathcal{C})$ *is an equivalence.*
2. *A functor* $F \in \mathrm{Fun}^{|-|}(\mathcal{D}^{-}_{\geqslant 0}(\mathcal{A}), \mathcal{C})$ *preserves finite coproducts if and only if its restriction to* $\mathcal{A}_{proj}$ *does.*

One last lemma

Part 1 is a corollary of the following:

Lemma (1.3.3.17)

The essential image of $\mathcal{D}^-_{\geqslant 0}(\mathcal{A})$ *in* $\mathrm{Fun}^\times(\mathcal{A}^{\mathrm{op}}_{proj}, \mathcal{S})$ *is the smallest full subcategory of* $\mathrm{Fun}(\mathcal{A}^{\mathrm{op}}_{proj}, \mathcal{S})$ *that contains the image of the Yoneda embedding and is closed under geometric realizations.*

Proof sketch.

The essential image contains the image of the Yoneda embedding.
▸ Lemma 1.3.3.16 and ▸ Lemma 1.3.3.11 imply that it is closed under geometric realizations.
Moreover, (the image of) every object $X \in \mathcal{D}^-_{\geqslant 0}(\mathcal{A})$ is equivalent to the geometric realization of the simplicial object in $\mathcal{A}_{\mathrm{proj}}$ that corresponds to (the chain complex underlying) X under the Dold–Kan correspondence. □

A more precise characterization

Theorem (1.3.3.8)

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Then

1. *The restriction functor* $\mathrm{Fun}^{|-|}(\mathcal{D}^{-}_{\geqslant 0}(\mathcal{A}), \mathcal{C}) \to \mathrm{Fun}(\mathcal{A}_{proj}, \mathcal{C})$ *is an equivalence.*
2. *A functor* $F \in \mathrm{Fun}^{|-|}(\mathcal{D}^{-}_{\geqslant 0}(\mathcal{A}), \mathcal{C})$ *preserves finite coproducts if and only if its restriction to* $\mathcal{A}_{proj}$ *does.*

Proof of part 2 of the more precise characterization

Proof sketch.

Let $F\colon \mathcal{D}^{-}_{\geqslant 0}(\mathcal{A}) \to \mathcal{C}$ be a functor that preserves geometric realizations such that $F' := F|\mathcal{A}_{\mathsf{proj}}$ preserves finite coproducts. We need to show that F preserves finite coproducts.

By possibly "extending" $\mathcal{C}$ by virtue of HTT 5.3.5.7, we may assume that it has all colimits. Then HTT 5.5.8.15 says that F', as a functor which preserves finite coproducts, extends to a functor in $\mathsf{Fun}^{\mathsf{cocont}}(\mathsf{Fun}^{\times}(\mathcal{A}^{\mathrm{op}}_{\mathsf{proj}}, \mathcal{S}), \mathcal{C})$.

Now restricting back to $\mathcal{D}^{-}_{\geqslant 0}(\mathcal{A}) \subseteq \mathcal{D}^{-}_{\geqslant 0}(\mathcal{A}^{\wedge}) \simeq \mathsf{Fun}^{\times}(\mathcal{A}^{\mathrm{op}}_{\mathsf{proj}}, \mathcal{S})$, we see that F itself also preserves finite coproducts. $\square$

A proof sketch for the main theorem

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\begin{array}{ccc}
\mathrm{Fun}^{\mathrm{rtex},\mathcal{A}_{\mathrm{proj}}\mapsto\mathcal{C}^{\heartsuit}}(\mathcal{D}^{-}(\mathcal{A}),\mathcal{C}) & & \\
\Big\updownarrow {\scriptstyle 1.3.3.11,\ 1.3.3.16} & & \\
\mathrm{Fun}^{|-|,\amalg,\mathcal{A}_{\mathrm{proj}}\mapsto\mathcal{C}^{\heartsuit}}(\mathcal{D}^{-}_{\geqslant 0}(\mathcal{A}),\mathcal{C}_{\geqslant 0}) & \xleftrightarrow{\ 1.3.3.8\ } & \mathrm{Fun}^{\amalg}(\mathcal{A}_{\mathrm{proj}},\mathcal{C}^{\heartsuit}) \\
 & & \Big\updownarrow {\scriptstyle 1.3.3.9\ \text{“projective resolutions”}} \\
 & & \mathrm{Fun}^{\mathrm{rex}_{\mathrm{ab}}}(\mathcal{A},\mathcal{C}^{\heartsuit})
\end{array}
$$